

FUNCTION WITH ITS FOURIER TRANSFORM SUPPORTED ON ANNULUS AND EIGENFUNCTION OF LAPLACIAN

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ABSTRACT. We explore the possibilities of reaching the characterization of eigenfunction of Laplacian as a degenerate case of the inverse Paley-Wiener theorem (characterizing functions whose Fourier transform is supported on a compact annulus) for the Riemannian symmetric spaces of noncompact type. Most distinguished prototypes of these spaces are the hyperbolic spaces. The statement and the proof of the main result work mutatis-mutandis for a number of spaces including Euclidean spaces and Damek-Ricci spaces.

1. INTRODUCTION

Let X be a rank one Riemannian symmetric space of noncompact type of dimension d , Δ be the Laplace-Beltrami operator on X induced by its Riemannian structure and B be its maximal distinguished boundary which is diffeomorphic to \mathbb{S}^{d-1} . A prototypical example of this class of spaces is the hyperbolic n -space. A representative result of this note is the following.

Theorem 1.1. *Suppose that for a nonzero function $f \in L^{2,\infty}(X)$, there are constants $c_1 \geq \rho^2, c_2 \leq 1/\rho^2$ such that*

$$\lim_{n \rightarrow \infty} \|\Delta^n f\|_{2,\infty}^{1/n} = c_1, \quad \lim_{n \rightarrow \infty} \|\Delta^{-n} f\|_{2,\infty}^{1/n} = c_2.$$

Let $\beta = \sqrt{1/c_2 - \rho^2}$ and $\alpha = \sqrt{c_1 - \rho^2}$. Then we have the following conclusions.

- (a) $c_1 c_2 \geq 1$.
- (b) *If $c_1 c_2 > 1$ then \tilde{f} is supported in the annulus $\mathbb{A}_\beta^\alpha = [\beta, \alpha] \times B$ around origin, but not inside any smaller annulus $\mathbb{A}_{\beta'}^{\alpha'}$, where $\beta < \beta'$ or $\alpha' < \alpha$.*
- (c) *If $c_1 c_2 = 1$ then $f = \mathcal{P}_\alpha F$ for some $F \in L^2(B)$, which is an eigenfunction of Δ with eigenvalue $-c_1$.*
- (d) *The annulus \mathbb{A}_β^α containing support of \tilde{f} may reduce to a ball around origin: $\mathbb{A}_0^\alpha = [0, \alpha] \times B$, but cannot collapse to the origin.*

(See Theorem 5.3 for a generalization and Proposition 5.1 for a related result of independent interest.)

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For a Schwartz class function g on X , \tilde{g} is an analogue of the Fourier transform on \mathbb{R}^n in polar coordinates and is known as geometric or the Helgason Fourier transform, defined on $\mathbb{R}^+ \times B$. In the statement above \tilde{f} is taken in the sense of tempered distribution. The Poisson transform \mathcal{P}_α is an analogue of the operator P_λ given by $P_\lambda F(x) = \int_{\mathbb{S}^{n-1}} F(y) e^{i\lambda x \cdot y} dy$ on \mathbb{R}^n . While P_λ maps a suitable function F on the boundary \mathbb{S}^{n-1} of \mathbb{R}^n to a function on \mathbb{R}^n , \mathcal{P}_α maps a function F defined on B to a function on X . Indeed $P_\lambda F$ or $\mathcal{P}_\alpha F$ are the basic eigenfunctions of the Laplacian of the corresponding spaces. In the hypothesis $\Delta^n f$ is used in the sense of distribution while $\Delta^{-n} f$ is in the sense of multiplier as spectrum of Δ on X is $[-\infty, -\rho^2]$ where ρ , the half-sum of positive roots, is realized as a positive number. We keep away from these interpretational worries, as we shall discuss them in details in Section 3. For other notation see Section 2.

An analogue of this theorem can be proved for \mathbb{R}^n replacing $L^{2,\infty}$ -norm by L^∞ -norm or by L^p -norm with $p > 2n/(n-1)$ for $n > 1$, to ensure the possibility of accommodating the eigenfunctions of Laplacian. One also obtains an analogue for the Heisenberg groups \mathbb{H}_n with L^∞ -norm in the hypothesis, and using the “Fourier transform” as defined in [40]. See [14, 22, 40], where some parts of these results for \mathbb{R}^n and \mathbb{H}_n are implicit. The situation in the Riemannian symmetric spaces of noncompact type, appears to be more intriguing, as indicated in [40] by constructing a counter example of Euclidean result for a complex hyperbolic space. We may point out here that the choice of the weak L^2 -norm (i.e. $L^{2,\infty}$ -norm) in the hypothesis is not at all arbitrary. Indeed, among all the Lorentz-norms (which, we recall include all L^p -norms), $L^{2,\infty}$ -norm is the unique option for X through which the theorem can accommodate the two possibilities (b) and (c) about the function f . We shall elaborate on these in Section 3 and cite some other “close to L^2 ” norms which can be used in place of weak L^2 -norm. Theorem 1.1 and its proof extends to the Damek-Ricci spaces (also called NA groups) which are Riemannian manifolds and solvable Lie groups but not in general symmetric spaces. Indeed rank one symmetric spaces of noncompact type accounts for a very small subset of all NA groups. However, we choose to illustrate the phenomenon only on rank one symmetric space, since, extending this to the set up of NA groups requires a lot of preliminaries, but the proof turns out to be the same (see Section 5 (3)).

To orient the readers we shall add some perspective of this study. An inverse Paley-Wiener theorem gives criterion on a function (with some integrability or regularity) which is necessary and sufficient for its Fourier transform to be compactly supported in a ball around origin, through the holomorphic extension of the function along with a growth condition on it. For Euclidean spaces it is same as the usual Paley-Wiener theorem. But for other spaces (e.g. a semisimple or nilpotent Lie group or

a symmetric space) where it is plausible to talk about Fourier transform, the usual and the inverse Paley-Wiener theorems are distinguished by the fact that domain of a function and its Fourier transform may be quite different and it is not at all clear where the complex analytic extension of the function has to be considered. For non-Euclidean spaces such inverse Paley-Wiener theorems are rather recent (see e.g. [13, 31, 25]). Very roughly, they state again that a suitable function with its Fourier transform compactly supported on its domain can be characterized from the holomorphic extension (in an appropriate domain) and growth of the function.

Unlike these results a *real* inverse Paley-Wiener theorem, does not consider and use the holomorphic extension of the inverse Fourier transform, but gives criterion involving norm estimates on the integral powers of Laplacian acting on the function. The main papers here are [7, 8, 42, 1, 2, 3, 9]. While most of these papers deal with Euclidean spaces, [1] considers the Riemannian symmetric space, where estimates on the L^2 -norm of positive integral powers of Laplacian is used. Part (b) of Theorem 1.1 is an extension of this as it characterizes functions whose Fourier transform is supported in a compact annulus around origin, under a weaker norm-condition. A different set of papers started with Roe [34] and followed by many, including [14, 21, 22, 40, 28, 33] try to characterize eigenfunctions of differential operators, in particular of the Laplacian, from a normed-estimate of a double sequence of functions $\{f_k\}$ related by $\Delta f_k = f_{k+1}$ for Δ of the space in context. Most of these papers deal with Euclidean spaces. One important exception is [40] where Strichartz establishes the failure of the Euclidean result for hyperbolic spaces, as mentioned above. But through [28] and [33] the result is restored for all Riemannian symmetric spaces of noncompact type (which includes hyperbolic spaces) and is also generalized to harmonic NA groups. A careful study reveals that the common thread between these two sets of results is the use of estimates of integral powers of Laplacian applied on the function. Our aim is to offer a version which accommodates both of these aspects.

We note in passing that ‘the compactly supported Fourier transform’ binds the real and the usual inverse Paley-Wiener theorem together, vindicating a relation between the estimates of $\Delta^n f$ and the regularity of f . Indeed the use of estimates of iterated action of Laplacian or more general operators on a function to retrieve regularity properties of the function is classical. We may cite for example Nelson, Kotake and Narasimhan [30, 24] and the references therein.

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2. PRELIMINARIES

In this section we shall establish notation and collect all ingredients to explain the statement and proof of the main result.

2.1. Generalities. For any $p \in [1, \infty)$, let $p' = p/(p-1)$. The letters \mathbb{N} , \mathbb{Z} , and \mathbb{R} , \mathbb{C} denote respectively the set of natural numbers, ring of integers, field real and complex numbers. We denote the nonzero real numbers, nonnegative real numbers and nonnegative integers respectively by \mathbb{R}^\times , \mathbb{R}^+ and \mathbb{Z}^+ . For $z \in \mathbb{C}$, $\Re z$, $\Im z$ and \bar{z} denote respectively the real and imaginary parts of z and the complex conjugate of z . For a set S in a topological space \bar{S} denotes its closure and for a set S in a measure space $|S|$ denotes its measure. We shall follow the standard practice of using the letters C, C_1, C_2, C' etc. for positive constants, whose value may change from one line to another. The constants may be suffixed to show their dependencies on important parameters. The notation $\langle f_1, f_2 \rangle$ for two functions or distributions f_1, f_2 , is frequently used in this article. It may mean $\int f_1 f_2$ when it makes sense. It may also mean that the distribution f_1 is acting on f_2 . Depending on the functions/distributions f_1, f_2 involved, the space could be X or its Fourier-dual $\mathbb{R}^+ \times B$, or \mathbb{R} with the canonical measures on them. As this notation is widely used in the literature, we hope this will not create any confusion. For two positive expressions f_1 and f_2 , by $f_1 \asymp f_2$ we mean that there are constants $C_1, C_2 > 0$ such that $C_1 f_1 \leq f_2 \leq C_2 f_1$.

2.2. Lorentz spaces. We shall briefly introduce Lorentz spaces (see [16, 39, 32] for details). Let (M, m) be a σ -finite measure space, $f : M \rightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|_{p,q}^* = \begin{cases} \left(\frac{q}{p} \int_0^\infty [f^*(t) t^{1/p}]^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t d_f(t)^{1/p} = \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty, \end{cases}$$

where for $\alpha > 0$, $d_f(\alpha) = |\{x \mid f(x) > \alpha\}|$, the distribution function of f and $f^*(t) = \inf\{s \mid d_f(s) \leq t\}$, the *decreasing rearrangement* of f . Let $L^{p,q}(M)$ be the set of all measurable $f : M \rightarrow \mathbb{C}$ such that $\|f\|_{p,q}^* < \infty$. We note the following.

- (i) The space $L^{p,\infty}(M)$ is known as the weak L^p -space.
- (ii) $L^{p,p}(M) = L^p(M)$ and $\|\cdot\|_{p,p}^* = \|\cdot\|_p$.
- (iii) For $1 < p, q < \infty$, the dual space of $L^{p,q}(M)$ is $L^{p',q'}(M)$ and the dual of $L^{p,1}(M)$ is $L^{p',\infty}(M)$.
- (iv) If $q_1 \leq q_2 \leq \infty$ then $L^{p,q_1}(M) \subset L^{p,q_2}(M)$ and $\|f\|_{p,q_2}^* \leq \|f\|_{p,q_1}^*$.

The Lorentz “norm” $\|\cdot\|_{p,q}^*$ is actually a quasi-norm and $L^{p,q}(M)$ is a quasi Banach space (see [16, p. 50]). For $1 < p \leq \infty$, there is an equivalent norm $\|\cdot\|_{p,q}$ which makes it a Banach space (see [39, Theorems 3.21, 3.22]). We shall slur over this difference and use the notation $\|\cdot\|_{p,q}$.

2.3. Symmetric space. We shall mostly use standard notation for objects related to semisimple Lie groups and the associated Riemannian symmetric spaces of noncompact type. Along with required preliminaries this can be found for example in [15, 18]. For making the article self-contained, we shall gather them without elaboration. We recall that a rank one Riemannian symmetric space of noncompact type (which we denote by X throughout this article) can be realized as a quotient space G/K , where G is a connected noncompact semisimple Lie group with finite centre and of real rank one and K a maximal compact subgroup of G . Thus $\mathbf{o} = \{K\}$ is the origin of X and a function on X can be identified with a function on G which is invariant under the right K -action. The group G acts naturally on $X = G/K$ by left translations $\ell_g : xK \rightarrow g^{-1}xK$ for $g \in G$. The Killing form on the Lie algebra \mathfrak{g} of G induces a G -invariant Riemannian structure and a G -invariant measure on X . Let Δ be the corresponding Laplace-Beltrami operator. For an element $x \in X$, let $|x| = d(x, \mathbf{o})$, where d is the distance associated to the Riemannian structure on X . Let \mathfrak{k} be the Lie algebra of K , $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition and \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Then $\dim \mathfrak{a} = 1$ as G is of real rank one. We denote the real dual of \mathfrak{a} by \mathfrak{a}^* . Let $\Sigma \subset \mathfrak{a}^*$ be the subset of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We recall that either $\Sigma = \{-\gamma, \gamma\}$ or $\{-2\gamma, -\gamma, \gamma, 2\gamma\}$ where γ is a positive root and the Weyl group W associated to Σ is $\{\text{Id}, -\text{Id}\}$ where Id is the identity operator. Let $m_\gamma = \dim \mathfrak{g}_\gamma$ and $m_{2\gamma} = \dim \mathfrak{g}_{2\gamma}$ where \mathfrak{g}_γ and $\mathfrak{g}_{2\gamma}$ are the root spaces corresponding to γ and 2γ . Then $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})\gamma$ denotes the half sum of the positive roots. Let H_0 be the unique element in \mathfrak{a} such that $\gamma(H_0) = 1$ and through this we identify \mathfrak{a} with \mathbb{R} as $t \mapsto tH_0$. Then $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \gamma(H) > 0\}$ is identified with the set of positive real numbers. We identify \mathfrak{a}^* and its complexification $\mathfrak{a}_\mathbb{C}^*$ with \mathbb{R} and \mathbb{C} by $t \mapsto t\gamma$ respectively $z \mapsto z\gamma$, $t \in \mathbb{R}$, $z \in \mathbb{C}$. By abuse of notation we will denote $\rho(H_0) = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$ by ρ . Let $\mathfrak{n} = \mathfrak{g}_\gamma + \mathfrak{g}_{2\gamma}$, $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}_+$ and $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$. Then N is a nilpotent Lie group and A is a one dimensional vector subgroup identified with \mathbb{R} . Precisely A is parametrized by $a_s = \exp(sH_0)$. The Lebesgue measure on \mathbb{R} induces a Haar measure on A by $da_s = ds$. Let M be the centralizer of A in K . The groups M and A normalizes N .

The group G has the Iwasawa decomposition $G = KAN$ and the polar decomposition $G = K\overline{A^+}K$. Through polar decomposition X is realized as $\overline{A^+} \times B$ where $B = K/M$ is the compact boundary of X . Using the Iwasawa decomposition $G = KAN$, we write an element $x \in G$ uniquely as $k(x) \exp H(x) n(x)$ where

$k(x) \in K, n(x) \in N$ and $H(x) \in \mathfrak{a}$. For $x \in X = G/K, b \in B = K/M$ let $A(x, b) = A(gK, kM) = -H(x^{-1}k)$. Let dg, dk and dm be the Haar measures of G, K and M respectively with $\int_K dk = 1$ and $\int_M dm = 1$. Let db be the normalized measure on $K/M = B$ induced by dk on K . We have the following integral formulae corresponding to the Iwasawa decompositions $G = KAN$ and the polar decomposition, which hold for any integrable function:

$$(2.1) \quad \int_G f(g)dg = C_1 \int_K \int_{\mathbb{R}} \int_N f(ka_t n) e^{2\rho t} dn dt dk,$$

and

$$(2.2) \quad \int_G f(g)dg = C_2 \int_K \int_0^\infty \int_K f(k_1 a_t k_2) (\sinh t)^{m_\gamma} (\sinh 2t)^{m_{2\gamma}} dk_1 dt dk_2,$$

The constants C_1, C_2 depend on the normalization of the Haar measures involved. Since $\sinh t \approx te^t/(1+t), t \geq 0$ it follows from (2.2) that

$$(2.3) \quad \begin{aligned} \int_G |f(g)|dg &\asymp C_3 \int_K \int_0^1 \int_K |f(k_1 a_t k_2)| t^{d-1} dk_1 dt dk_2 \\ &+ C_4 \int_K \int_1^\infty \int_K |f(k_1 a_t k_2)| e^{2\rho t} dk_1 dt dk_2 \end{aligned}$$

where $d = m_\gamma + m_{2\gamma} + 1$ is the dimension of the symmetric space. For a integrable function f on X , $\int_G f(g)dg = \int_X f(x)dx$ where in the left hand side f is considered as a right K -invariant function on G and dg is the Haar measure on G , while on the right side dx is the G -invariant measure on X .

2.3.1. Poisson transform. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel: $x \mapsto e^{-(i\lambda+\rho)H(x^{-1})}$ is an eigenfunction of the Laplace Beltrami operator Δ with eigenvalue $-(\lambda^2 + \rho^2)$. For any $\lambda \in \mathbb{C}$ and $F \in L^1(B)$ we define the Poisson transform \mathcal{P}_λ of F by (see [18, p. 279]) by

$$\mathcal{P}_\lambda F(x) = \int_B F(b) e^{(i\lambda+\rho)A(x,b)} db \quad \text{for } x \in X.$$

Then,

$$\Delta \mathcal{P}_\lambda F = -(\lambda^2 + \rho^2) \mathcal{P}_\lambda F.$$

A function f on X is left K -invariant or *radial* if $f(kx) = f(x)$ for all $k \in K$ and $x \in X$. Note that a left K -invariant function on X can be identified with a K -biinvariant function on G . We shall use both the terms radial and K -biinvariant for such functions. For any function space $\mathcal{L}(X)$, by $\mathcal{L}(G//K)$ we mean its subset of K -biinvariant functions. For a suitable function f on X we define its *radialization* Rf by $Rf(x) = \int_K f(kx)dk$. It is clear that Rf is a radial function and if f is radial then $Rf = f$. We also note that for (i) $\phi, \psi \in C_c^\infty(X)$, $\langle R\phi, \psi \rangle = \langle \phi, R\psi \rangle$ and (ii) $R(\Delta\phi) = \Delta(R\phi)$. From (i) it follows that $\int_X f(x)dx = \int_X Rf(x)dx$ and

hence $\|Rf\|_1 \leq \|f\|_1$. Interpolating [39, p. 197] with the trivial L^∞ -boundedness of the operator R we get

$$\|Rf\|_{p,q} \leq \|f\|_{p,q} \text{ for } 1 < p < \infty, 1 \leq q \leq \infty.$$

For any $\lambda \in \mathbb{C}$ the elementary spherical function ϕ_λ is given by,

$$\phi_\lambda(x) = \mathcal{P}_\lambda 1(x) = \int_K e^{-(i\lambda + \rho)H(xk)} dk = \int_K e^{(i\lambda - \rho)H(xk)} dk \text{ for all } x \in G,$$

where by 1 we denote the constant function 1 on $B = K/M$. Hence $\Delta\phi_\lambda = -(\lambda^2 + \rho^2)\phi_\lambda$ for $\lambda \in \mathbb{C}$. It follows that for $\lambda \in \mathbb{C}$, ϕ_λ is radial, $\phi_\lambda = \phi_{-\lambda}$ and it satisfies the following estimates: (see [6], [15, (4.6.5)])

$$(2.4) \quad \begin{aligned} |\phi_{\alpha + i\gamma_p \rho}(x)| &\asymp e^{-(2\rho/p')|x|}, \quad \alpha \in \mathbb{R}, 0 < p < 2, \gamma_p = 2/p - 1; \\ |\phi_0(a_t)| &\leq Ce^{\rho t}(1 + |t|), \text{ for } t > 0 \end{aligned}$$

and

$$(2.5) \quad \left| \frac{d^n}{d\lambda^n} \phi_\lambda(x) \right| \leq C(1 + |x|)^n \phi_{\Im\lambda}(x) \text{ for } \lambda \in \mathbb{C}.$$

2.3.2. Spherical Fourier Transform. For a measurable function f of X , we define its *spherical Fourier transform* \widehat{f} and its inverse as follows (see [18, p. 425, p. 454]),

$$\widehat{f}(\lambda) = \int_X f(x) \phi_{-\lambda}(x) dx, \quad \lambda \in \mathfrak{a}^*, \quad f(x) = C \int_{\mathfrak{a}^*} \widehat{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda,$$

whenever the integrals make sense. Here $c(\lambda)$ is the Harish-Chandra c -function, $d\lambda$ is the Lebesgue measure on $\mathfrak{a}^* \equiv \mathbb{R}$ and $|c(\lambda)|^{-2} d\lambda$ is the spherical Plancherel measure on \mathfrak{a}^* and C is a normalizing constant. Since $\phi_\lambda = \phi_{-\lambda}$ we have $\widehat{f}(\lambda) = \widehat{f}(-\lambda)$, hence we can consider \widehat{f} as a function on \mathbb{R}^+ .

2.3.3. Helgason Fourier Transform. For a function f on X , its Helgason Fourier transform (or Fourier transform) is defined by

$$\widetilde{f}(\xi, b) = \int_X f(x) e^{(-i\xi + \rho)(A(x, b))} dx$$

for $\xi \in \overline{\mathfrak{a}_+^*} \equiv \mathbb{R}^+$, $b \in B$ for which the integral exists. (See [19, pp. 199-203] for details.) The Fourier transform $f(x) \rightarrow \widetilde{f}(\xi, b)$ extends to an isometry of $L^2(X)$ onto $L^2(\mathbb{R}^+ \times B, |c(\xi)|^{-2} d\xi db)$ and we have,

$$\int_X f_1(x) \overline{f_2(x)} dx = C \int_{\mathbb{R}^+ \times B} \widetilde{f}_1(\xi, b) \overline{\widetilde{f}_2(\xi, b)} |c(\xi)|^{-2} d\xi db.$$

For functions f, g on X with g radial, $\widetilde{g}(\xi, k) = \widehat{g}(\xi)$ and $\widetilde{f * g}(\xi, b) = \widetilde{f}(\xi, b) \widehat{g}(\xi)$ for $\xi \in \mathbb{C}$ and $b \in B$ whenever the quantities $f * g, \widetilde{f * g}, \widetilde{f}$ and \widehat{g} make sense.

2.3.4. *Schwartz spaces, tempered distributions.* For $1 \leq p \leq 2$, the L^p -Schwartz space $C^p(X)$ is defined (see [5]) as the set of C^∞ -functions on X such that

$$\gamma_{r,D}(f) = \sup_{x \in S} |Df(x)| \phi_0^{-2/p} (1 + |x|)^r < \infty,$$

for all nonnegative integers r and left invariant differential operators D on X . We topologize $C^p(X)$ by the seminorms $\gamma_{r,D}$. Then $C^p(X)$ is a dense subset of $L^p(X)$. Let $C^p(G//K)$ be the set of radial functions in $C^p(X)$. We shall primary use $C^2(X)$, the L^2 -Schwartz space. Let $C^2(\hat{X})$ (respectively $C^2(\widehat{G//K})$) be the image of $C^2(X)$ (respectively of $C^2(G//K)$) under $f \mapsto \tilde{f}$ (respectively $f \mapsto \hat{f}$). Then (see [5]) $f \mapsto \hat{f}$ is a topological isomorphism from $C^2(G//K)$ to $C^2(\widehat{G//K}) = S(\mathbb{R})_{\text{even}}$ where $S(\mathbb{R})$ is the set of Schwartz class functions on \mathbb{R} , and $S(\mathbb{R})_{\text{even}}$ denotes the subspace of even functions in $S(\mathbb{R})$. We do not need the explicit description of $C^2(\hat{X})$, for which along with the isomorphism of $f \mapsto \tilde{f}$ from $C^2(X)$ to $C^2(\hat{X})$ we refer to [12, Theorem 4.8.1].

We denote the dual space of $C^p(G//K)$ (respectively $C^p(X)$) by $C^p(G//K)'$ (respectively $C^p(X)'$). Elements of $C^p(G//K)'$ and $C^p(X)'$ are called respectively the K -bi-invariant L^p -tempered distributions and L^p -tempered distributions on X . It is clear that $L^{p'}(G//K) \subset C^p(G//K)'$ and $L^{p'}(X) \subset C^p(X)'$ for $1 \leq p \leq 2$. For an L^2 -tempered distribution \tilde{f} , \tilde{f} is defined as a continuous linear functional on $C^2(\hat{X})$: for $\phi \in C^2(X)$, $\langle \tilde{f}, \tilde{\phi} \rangle = \langle f, \phi \rangle$.

For a function $\phi \in C^2(X)$, we define support of $\tilde{\phi}$ as a subset of $\mathbb{R}^+ \times B$ by

$$\text{Suppt } \tilde{\phi} = \overline{\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \tilde{\phi}(\lambda, b) \neq 0\}},$$

If ϕ is also K -biinvariant then $\tilde{\phi}(\lambda, b) = \hat{\phi}(\lambda)$ for all $b \in B$ and hence $\text{Suppt } \tilde{\phi} = \overline{\{\lambda \in \mathbb{R}^+ \mid \hat{\phi}(\lambda) \neq 0\}} \times B$. When ϕ is K -biinvariant, by abuse of terminology, the set $\{\lambda \in \mathbb{R}^+ \mid \hat{\phi}(\lambda) \neq 0\}$ will also be called support of ϕ . We recall that $L^{2,\infty}(X) \subset C^2(X)'$ (see Proposition 3.2 (ii) below). For a function $f \in L^{2,\infty}(X)$, the distributional support of \tilde{f} is the complement of the largest open set $U \subset \mathbb{R}^+ \times B$ such that for any $\phi \in C^2(X)$ with $\text{Suppt } \tilde{\phi}$ contained in U , $\langle f, \phi \rangle = 0$.

If for a function $f \in L^{2,\infty}(X)$, $\text{Suppt } \tilde{f}$ is an empty set then $f \equiv 0$. Indeed, $\text{Suppt } \tilde{f}$ is empty implies that f annihilates all functions in $C^2(X)$ and hence it is zero as a L^2 -tempered distribution.

2.3.5. *Abel transform.* For a radial function f on X its Abel transform $\mathcal{A}f$ is defined by:

$$\mathcal{A}f(a) = e^{\rho(\log a)} \int_N f(an) dn, \text{ for } a \in A,$$

whenever the integral makes sense. Through the identification of A with \mathbb{R} we can write it as:

$$\mathcal{A}f(t) = e^{\rho t} \int_N f(a_t n) dn \text{ for } t \in \mathbb{R}.$$

For $f \in S(\mathbb{R})$ let $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$ be its Euclidean Fourier transform at $\xi \in \mathbb{R}$.

We recall: (see [5]) (a) (*slice projection theorem*) for any $f \in C^2(G//K)$, $\lambda \in \mathbb{R}$, $\mathcal{F}(\mathcal{A}f)(\lambda) = \widehat{f}(\lambda)$, (b) $\mathcal{A} : C^2(G//K) \rightarrow S(\mathbb{R})_{\text{even}}$ is a topological isomorphism. By duality from the second statement we get that the adjoint of the Abel transform $\mathcal{A}^* : S(\mathbb{R})'_{\text{even}} \rightarrow C^2(G//K)'$ is an isomorphism (see [20, p. 541]).

3. SOME PREPARATORY DISCUSSIONS

In this section we shall explain the statement of the main result, highlight some of its features and gather some results which will be used in the next section.

(1) As mentioned in the introduction, the weak L^2 -norm in the hypothesis is the only possible Lorentz norm for the formulation. We shall elaborate on this.

As the statement of Theorem 1.1 involves Fourier transform, tempered distribution is a natural choice to work with. An $L^{2,\infty}$ -function on X is an L^2 -tempered distribution and the space $L^{2,\infty}(X)$ is *close to* $L^2(X)$, where usually the inverse Paley-Wiener theorems are stated. We recall that for $1 \leq q < \infty$, $L^{2,q}(X) \subset C^2(X)'$ (see Proposition 3.2 (ii) below), i.e. an $L^{2,q}$ -function is also an L^2 -tempered distribution. But $L^{2,q}$ -norms (which in particular includes $L^2 = L^{2,2}$ -norm) discards the possibility of f being an eigenfunction (see Proposition 3.2 (vi) below). Hence in this case $c_1 c_2 > 1$.

Suppose that we take $f \in L^{p,q}(X)$ with $1 \leq p < 2, 1 \leq q \leq \infty$ and use $L^{p,q}$ -norm in the hypothesis instead of $L^{2,\infty}$ -norm. Then again f is an L^2 -tempered distribution. Indeed $C^2(X) \subset L^2(X) \cap L^\infty(X)$ and hence $C^2(X) \subset L^{p',q}(X)$ for p, q in the range above by interpolation. This implies by duality that $L^{p,q'}(X) \subset C^2(X)'$. But Fourier transform $\widetilde{f}(\lambda, b)$ of such a function f which exists point-wise, has complex-analytic extension in λ in a strip for almost every $b \in B$ (see [29, 32]) and so if the limits in the hypothesis exist, the only possibilities are $c_1 = \infty$ and $c_2 = \rho^{-2}$, i.e. the annulus $\mathbb{A}_\beta^\alpha = \mathbb{C} \times B$.

Lastly if $f \in L^{p,q}(X)$ with $p > 2, 1 \leq q \leq \infty$, then f is an $L^{p'}$ -tempered distribution where $p' < 2$ (and in general not an L^2 -tempered distribution). See [28, section 6]. It is clear that the usual definition of distributional support of its Fourier transform is not meaningful for such a function since there is no function in $C^{p'}(X)$ whose Fourier transform is compactly supported. On the other hand there are functions

$f \in L^{p,q}(X)$ satisfying

$$\lim_{n \rightarrow \infty} \|\Delta^n f\|_{p,q}^{1/n} = c_1, \quad \lim_{n \rightarrow \infty} \|\Delta^{-n} f\|_{p,q}^{1/n} = 1/c_1$$

which are not eigenfunctions (not even generalized eigenfunctions) of Δ . An easy example is the following. We take two points $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $|\Im \lambda_i| < |2/p - 1|\rho$ and $|\lambda_1^2 + \rho^2| = |\lambda_2^2 + \rho^2| = \delta$ for some fixed $\delta > (4\rho^2)/(pp')$. Indeed uncountably many $\lambda \in \mathbb{C}$ satisfy this for any such fixed δ . Then it is easy to verify that if $f = \phi_{\lambda_1} + \phi_{\lambda_2}$ then f is not a generalized eigenfunction but satisfies the hypothesis of Theorem 1.1 with the substitution of $L^{2,\infty}$ -norm by $L^{p,q}$ -norm for p, q as above.

(2) Outside the set of Lorentz norms and L^p -norms there are some prominent size estimates which are used in the literature to characterize eigenfunctions of Laplacian as Poisson transforms. We shall mention only two of them. Let $B(0, r) = \{x \in X \mid |x| < r\}$ be the geodesic ball of radius r . For $1 < p < \infty$, $1 \leq q < \infty$ and a function f on X we define

$$(3.1) \quad M_p(f) = \left(\limsup_{r \rightarrow \infty} \frac{1}{r} \int_{B(0,r)} |f(x)|^p dx \right)^{1/p},$$

$$(3.2) \quad \mathcal{K}_{p,q}(f) = \|\mathcal{K}_q(f)\|_{p,\infty}, \text{ where } \mathcal{K}_q(f)(x) = \left(\int_K |f(kx)|^q dk \right)^{1/q}.$$

Any function f on X satisfying $M_2(f) < \infty$ or $\mathcal{K}_{2,q}(f) < \infty$ is an L^2 -tempered distribution. (See the line above Section 4). Since the argument in the proof of Theorem 1.1 works under the assumption that f is an L^2 -tempered distribution, we can substitute $L^{2,\infty}$ -norm by M_2 -norm or by $\mathcal{K}_{2,q}$ -norm. See [28] for the background relevant to these norms.

(3) Negative powers of Δ used in the statement of Theorem 1.1 can be interpreted in terms of radial multipliers. Precisely, Δ^{-1} is an L^p -multiplier for $1 < p < 2$ (see [4]) and hence an $L^{p'}$ -multiplier. Hence by interpolation [39, p. 197] defines a bounded operator from $L^{2,\infty}(X)$ to itself. This is a benefit of the fact that in X (and NA groups) the spectrum of Δ does not contain 0 (see [41]). But keeping in mind the spaces (e.g. \mathbb{R}^n) where this interpretation is not valid, we can have an alternative formulation following [14, 40, 22], which in our case is only a change of notation.

Theorem 3.1. *Let $\{f_k\}_{k \in \mathbb{Z}}$ be a doubly infinite sequence of nonzero functions in $L^{2,\infty}(X)$ with $\Delta f_k = f_{k+1}$ for all $k \in \mathbb{Z}$. Suppose for constants $c_1 \geq \rho^2, c_2 \leq 1/\rho^2$,*

$$\lim_{k \rightarrow \infty} \|f_k\|_{2,\infty}^{1/k} = c_1, \quad \lim_{k \rightarrow \infty} \|f_{-k}\|_{2,\infty}^{1/k} = c_2.$$

Then we have the conclusions of Theorem 1.1 for $f = f_0$.

Indeed the substitution $f = f_0$ and $f_k = \Delta^k f_0 = \Delta^k f$ for $k \in \mathbb{Z}$ reduces the hypothesis of this theorem to that of Theorem 1.1.

(4) We recall that Δ^n for $n \in \mathbb{N}$ commutes with translations, precisely $\Delta^n \ell_x f = \ell_x \Delta^n f$ for any $x \in G$ and a locally integrable function f on X . It is also not difficult to see that $\Delta^{-n} \ell_x f = \ell_x \Delta^{-n} f$ for any $n \in \mathbb{N}$. Similarly it can be verified that Δ^n for $n \in \mathbb{Z}$ commutes with the radialization operator R , i.e. $\Delta^n(R(f)) = R(\Delta^n f)$.

(5) We conclude this section collecting a few not-so-well-known results, some of which are used in the discussion above and some will be required for the main argument.

- Proposition 3.2.** (i) $C^2(X)$ is a dense subset of $L^{2,1}(X)$ and there exists a seminorm ν of $C^2(X)$ such that for all $\phi \in C^2(X)$, $\|\phi\|_{2,1} \leq C\nu(\phi)$.
- (ii) For $f \in L^{2,\infty}(X)$, there exists a seminorm ν of $C^2(X)$ such that for all $\phi \in C^2(X)$, $|\langle f, \phi \rangle| \leq C\|f\|_{2,\infty}\nu(\phi)$. That is $f \in L^{2,\infty}(X)$ is an L^2 -tempered distribution. Since for any $q < \infty$, $L^{2,q}(X) \subset L^{2,\infty}(X)$ and $\|f\|_{2,\infty} \leq \|f\|_{2,q}$, any $f \in L^{2,q}(X)$ is also an L^2 -tempered distribution.
- (iii) Let $1 \leq q \leq \infty$ be fixed. If for a nonnegative radial measure μ on X , $\widehat{\mu}(0) < \infty$, then $T_\mu : f \rightarrow f * \mu$ defines a bounded operator from $L^{2,q}$ to itself and the operator norm satisfies $\|T_\mu\|_{L^{2,q} \rightarrow L^{2,q}} \leq \widehat{\mu}(0)$.
- (iv) For $f \in L^{2,\infty}(X)$ and $\psi \in C^2(G/K)$, $\|f * \psi\|_{2,\infty} \leq \|f\|_{2,\infty}\nu(\psi)$ for some seminorm ν of $C^2(X)$.
- (v) If a nonzero function f on X satisfies $\Delta f = -\rho^2 f$, then $f \notin L^{2,\infty}(X)$. In particular $\phi_0 \notin L^{2,\infty}(X)$.
- (vi) If a nonzero function f on X satisfies $\Delta f = -(\lambda^2 + \rho^2)f$, for some $\lambda \in \mathbb{R}^\times$, then $f \notin L^{2,q}(X)$ for any $q < \infty$.
- (vii) For any $\lambda \in \mathbb{R}^\times$, $\phi_\lambda \in L^{2,\infty}(X)$.
- (viii) Suppose that a function f on X satisfies $\Delta f = -(\lambda^2 + \rho^2)f$ with $\lambda \in \mathbb{R}^\times$. Then $f = \mathcal{P}_\lambda u$ for some $u \in L^2(B)$ if and only if $f \in L^{2,\infty}(X)$ and in that case $\|\mathcal{P}_\lambda u\|_{2,\infty} \leq C_\lambda \|u\|_{L^2(B)}$.

Proof. (i) follows from the definition of $C^2(X)$ and the fact that for an appropriately large M , the function $\phi_0(x)(1 + |x|)^{-M} \in L^{2,1}(X)$. See [28, Lemma 6.1.1]. Denseness of $C^2(X)$ is a consequence of denseness of $C_c^\infty(X)$ in $L^{2,1}(X)$. (ii) is immediate from (i) and Hölder's inequality. See also [28, Lemma 6.1.1]. (iii) is a particular case of a more general result proved in [36, Lemma 3.2.1] and [6]. For (iv) we have

$$|\widehat{\psi}|(0) = \int_X |\psi(x)|\phi_0(x)dx \leq \sup_{x \in X} [|\psi(x)|\phi_0^{-1}(x)(1 + |x|)^M] \int_X \phi_0^2(x)(1 + |x|)^{-M} dx.$$

It follows from the estimate of ϕ_0 and the measure on X (see Section 2) that $C = \int_X \phi_0^2(x)(1+|x|)^{-M} dx < \infty$ for suitably large M . We define

$$\nu(\psi) = \sup_{x \in X} [|\psi(x)|\phi_0^{-1}(x)C(1+|x|)^M]$$

to get $\widehat{|\psi|}(0) \leq \nu(\psi)$. Thus by (iii),

$$\|f * \psi\|_{2,\infty} \leq \| |f| * |\psi| \|_{2,\infty} = \|T_{|\psi|}(|f|)\|_{2,\infty} \leq \|f\|_{2,\infty} \widehat{|\psi|}(0) = \|f\|_{2,\infty} \nu(\psi).$$

For (v), (vi), (vii) and (viii) we refer to [28, Proposition 3.1.1, (2.2.6) and Theorem 4.3.5] and [26]. ((vii) is also a particular case of (viii).) \square

For the corresponding results in particular that of (i), (ii) and (viii) above for M_2 norm and $\mathcal{K}_{2,q}$ norm, we refer to [28, Lemma 6.1.1] and [11, 23].

4. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 1.1. We begin with a few observations and results which relate the support of the Fourier transform of a function on X with the support of the Fourier transform of its translation and radialization.

Proposition 4.1. *Let $g \in C^2(X)$ and $\lambda \in \mathbb{R}^+$. Then $(\lambda, b) \in \text{Suppt } \widetilde{g}$ for some $b \in B$ if and only if $\lambda \in \text{Suppt } \widehat{R(\ell_x g)}$ for some $x \in G$.*

Proof. Note that for $\lambda \in \mathbb{R}$ (see [19, p. 200]),

$$\widehat{R(\ell_x g)}(\lambda) = \widehat{\ell_x g}(\lambda) = g * \phi_\lambda(x^{-1}) = \int_B \widetilde{g}(\lambda, b) e^{(i\lambda + \rho)(A(x^{-1}, b))} db = \mathcal{P}_\lambda \widetilde{g}(\lambda, \cdot)(x^{-1}),$$

where in the last equality above we have considered $\widetilde{g}(\lambda, \cdot)$ as a function on B . If $(\lambda, b) \notin \text{support } \widetilde{g}$ for all $b \in B$ then clearly $\lambda \notin \text{support } \widehat{R(\ell_x g)}$ for all $x \in G$. Conversely, if $\lambda \notin \text{support } \widehat{R(\ell_x g)}$ for all $x \in G$, then $\mathcal{P}_\lambda \widetilde{g}(\lambda, \cdot) \equiv 0$. Using simplicity criterion ([19, pp. 152, 165]) this implies that $\widetilde{g}(\lambda, \cdot) \equiv 0$. \square

Proposition 4.2. *Let $g \in C^2(X)$. If support of \widetilde{g} intersects the sphere $\{\gamma\} \times B$ for some $\gamma \geq 0$, then for any $y \in G$, support of $\widetilde{\ell_y g}$ also intersects $\{\gamma\} \times B$.*

Proof. We have

$$\widetilde{\ell_y g}(\xi, kM) = \int_X g(y^{-1}x) e^{(i\xi - \rho)H(x^{-1}k)} dx.$$

With the substitution $y^{-1}x = z$ and using the identity $H(z^{-1}y^{-1}k) = H(y^{-1}k) + H(z^{-1}K(y^{-1}k))$ ([19, p.200]) we get from above

$$\begin{aligned} \widetilde{\ell_y g}(\xi, kM) &= [e^{(i\xi - \rho)H(y^{-1}k)}] \int_X g(z) e^{(i\xi - \rho)H(z^{-1}K(y^{-1}k))} dz \\ &= [e^{(i\xi - \rho)H(y^{-1}k)}] \widetilde{g}(\xi, K(y^{-1}k)). \end{aligned}$$

Suppose that $\tilde{g}(\gamma, b) \neq 0$ for $b = k_1 M$. Let $K(yk_1) = k$. Then $K(y^{-1}k) = k_1$ and hence $\tilde{\ell}_y g(\gamma, kM) \neq 0$, which proves the assertion. \square

We note that for Theorem 1.1, it is required to find only the inner and outer radii of the support of \tilde{f} . Precisely, outer and inner radii of support of \tilde{f} are α and β respectively if support of \tilde{f} is contained in the annulus $[\beta, \alpha] \times B$ but not contained in $[\beta', \alpha'] \times B$ when $\beta < \beta'$ or $\alpha' < \alpha$.

Observation 4.3. *Let $f \in L^{2,\infty}(X)$. Then the radii of support of \tilde{f} are the same as the radii of support of $\ell_x f$ for any $x \in G$. Suppose that radii of support of \tilde{f} are α, β . We take a function $g \in C^2(G/K)$, such that $\text{Suppt } \tilde{g}$ is contained in $\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda > \alpha\}$. Then by Proposition 4.2, $\text{Suppt } \tilde{\ell_{x^{-1}} g}$ for any $x \in G$, is also contained in $\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda > \alpha\}$. Hence $\langle f, \ell_{x^{-1}} g \rangle = 0$. Therefore $\langle \ell_x f, g \rangle = \langle f, \ell_{x^{-1}} g \rangle = 0$. Since f is a translation of $\ell_x f$, outer radius of support of $\ell_x f$ is same with outer radius of support of \tilde{f} . Similarly we can show that inner radius of \tilde{f} and of $\ell_x f$ are same.*

Observation 4.4. *Let $f \in L^{2,\infty}(X)$. Suppose that $\text{Suppt } \tilde{f} \subset \{\alpha\} \times B$. Then $\text{Suppt } \widehat{R(\ell_x f)} \subset \{\alpha\}$ for any $x \in G$. Indeed if $R(\ell_x f) = 0$ we have nothing to show. So we assume $R(\ell_x f) \neq 0$. We take a function $g \in C^2(X)$ with $\text{Suppt } \tilde{g} \subset \{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda \neq \alpha\}$. By Proposition 4.1, $\text{Suppt } \widehat{R(g)} \subset \{\lambda \in \mathbb{R}^+ \mid \lambda \neq \alpha\} \times B$. Therefore by Observation 4.3, $\langle \ell_x f, Rg \rangle = 0$ and hence $\langle R(\ell_x f), g \rangle = \langle \ell_x f, Rg \rangle = 0$.*

This makes us ready to present the proof of the main result. For readers' convenience distinguished parts of the proof of (c) are separated as a series of lemmas, given after the proof of this theorem. Lemma 4.10 (and its generalization Proposition 5.1 in the next section) may be of independent interest.

Proof of Theorem 1.1. We shall prove (b) and (c) and then use them to prove (a) and (d).

(b) We take $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that $\alpha < \lambda_1 < \lambda_2$. Let $\phi \in C^2(\widehat{G//K})$ be supported on $[\lambda_1, \lambda_2]$. We claim that $\langle \tilde{f}, \phi \rangle = 0$.

Let $\epsilon = \frac{1}{4}(\lambda_1^2 - \alpha^2) > 0$ where $\alpha = \sqrt{c_1 - \rho^2}$. From the hypothesis we know that there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$(4.1) \quad \left| \|\Delta^n f\|_{2,\infty}^{1/n} - c_1 \right| < \epsilon \text{ and hence } (c_1 - \epsilon)^n < \|\Delta^n f\|_{2,\infty} < (c_1 + \epsilon)^n.$$

As $\widetilde{\Delta^k f} = (-1)^k (\lambda^2 + \rho^2)^k \widetilde{f}$ (where λ is a dummy variable),

$$\begin{aligned} |\langle \widetilde{f}, \phi \rangle| &= |\langle \widetilde{\Delta^k f}, \frac{1}{(\lambda^2 + \rho^2)^k} \phi \rangle| \\ &= |\langle \Delta^k f, \psi_k \rangle| \\ &\leq \|\Delta^k f\|_{2,\infty} \|\psi_k\|_{2,1} \\ &\leq \|\Delta^k f\|_{2,\infty} \nu(\psi_k) \\ &\leq \|\Delta^k f\|_{2,\infty} \mu(\widehat{\psi_k}) \end{aligned}$$

where $\psi_k \in C^2(G//K)$ is the inverse spherical transform of $(\lambda^2 + \rho^2)^{-k} \phi \in C^2(\widehat{G//K})$ and ν, μ are seminorms of $C^2(X)$ and of $C^2(\widehat{X})$ respectively. We have used above Hölder's inequality, that $\|\psi_k\|_{2,1} \leq \nu(\psi_k)$ (Proposition 3.2 (i)) and the isomorphism between $C^2(G//K)$ and $C^2(\widehat{G//K})$ (see subsection 2.3.4).

Thus for $k \geq N$, we have

$$(4.2) \quad |\langle \widetilde{f}, \phi \rangle| \leq (c_1 + \epsilon)^k \mu(\widehat{\psi_k}) = \mu \left[\left(\frac{\alpha^2 + \rho^2 + \epsilon}{\lambda^2 + \rho^2} \right)^k \phi \right].$$

Recall that ϕ is supported on $[\lambda_1, \lambda_2]$. For $\lambda \in [\lambda_1, \lambda_2]$ and the ϵ chosen above,

$$\lambda^2 + \rho^2 \geq \lambda_1^2 + \rho^2 = \alpha^2 + \rho^2 + 4\epsilon > \alpha^2 + \rho^2 + \epsilon.$$

Hence given any $\delta > 0$ we can find $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for $k \geq N_1$, $\mu[\dots] < \delta$ in (4.2) and hence $|\langle \widetilde{f}, \phi \rangle| < \delta$. This establishes the claim and proves that f annihilates any function $\phi \in C^2(G//K)$ such that $\widehat{\phi}$ is supported in a compact set of \mathbb{R}^+ outside $[0, \alpha]$.

A step by step adaptation of this argument will show that f also annihilates any function $\psi \in C^2(G//K)$ such that $\widehat{\psi}$ is supported in a compact set of \mathbb{R}^+ outside $[\beta, \infty)$. We include a sketch. We take ξ_1, ξ_2 with $0 < \xi_1 < \xi_2 < \beta$. Let $\phi \in C^2(\widehat{G//K})$ be supported on $[\xi_1, \xi_2]$. We need to show that $\langle \widetilde{f}, \phi \rangle = 0$. We take

$$(4.3) \quad \epsilon = \frac{\beta^2 - \xi_2^2}{4(\xi_2^2 + \rho^2)(\beta^2 + \rho^2)} > 0.$$

It follows from the hypothesis that there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$(4.4) \quad \|\Delta^{-n} f\|_{2,\infty}^{1/n} - c_2 < \epsilon \text{ and hence } (c_2 - \epsilon)^n < \|\Delta^{-n} f\|_{2,\infty} < (c_2 + \epsilon)^n.$$

Following steps of the previous part of the proof we get

$$|\langle \widetilde{f}, \phi \rangle| = |\langle \widetilde{\Delta^{-k} f}, (\lambda^2 + \rho^2)^k \phi \rangle| \leq \|\Delta^{-k} f\|_{2,\infty} \mu(\widehat{\psi_k})$$

where $\psi_k \in C^2(G//K)$ is the inverse image of $(\lambda^2 + \rho^2)^k \phi \in C^2(\widehat{G//K})$ and μ is a seminorm of $C^2(\widehat{X})$. Taking $k \geq N$, we have

$$|\langle \widetilde{f}, \phi \rangle| \leq (c_2 + \epsilon)^k \mu(\widehat{\psi_k}) = \mu \left[\left(\frac{1}{\beta^2 + \rho^2} + \epsilon \right)^k (\lambda^2 + \rho^2)^k \phi \right].$$

Since ϕ is supported on $[\xi_1, \xi_2]$, by (4.3) we have for $\lambda \in [\xi_1, \xi_2]$,

$$4\epsilon + \frac{1}{\beta^2 + \rho^2} = \frac{1}{\xi_2^2 + \rho^2} \leq \frac{1}{\lambda^2 + \rho^2}.$$

The rest of the argument is same as the first part.

We have shown that f annihilates any function $\psi \in C^2(G//K)$ with $\widehat{\psi}$ compactly supported outside $[\beta, \alpha]$. We shall now remove the condition of K -biinvariantness from ϕ . By Observation 4.3, for any $x \in G$, $\ell_x f$ also annihilates all $\psi \in C^2(G//K)$ for which $\widehat{\psi}$ is compactly supported outside $[\beta, \alpha]$. Since $\psi(x) = \psi(x^{-1})$, this implies that $f * \psi(x) = 0$ for all $x \in G$. Noting that $f * \psi \in L^{2,\infty}(X)$ (Proposition 3.2 (iv)) we have for any $g \in C^2(X)$, $\langle f * \psi, g \rangle = 0$ and hence by Fubini's theorem $\langle f, g * \psi \rangle = 0$.

We take $g \in C^2(X)$ with $\text{Suppt } \widetilde{g}$ contained in an open set $U \subset \mathbb{R}^+ \times B$ such that $([\beta, \alpha] \times B) \cap U = \emptyset$. We find another open set $U_1 \subset \mathbb{R}^+ \times B$ satisfying $U \subset U_1$, U_1 is B -invariant (i.e. if $(\lambda, b) \in U_1$ for some $b \in B$, then $\{\lambda\} \times B \subset U_1$) and $([\beta, \alpha] \times B) \cap U_1 = \emptyset$. We take a $\psi \in C^2(G//K)$ such that $\widehat{\psi}$ is supported on U_1 and $\widehat{\psi} \equiv 1$ on U (hence on the set $\{\lambda \mid (\lambda, b) \in U \text{ for some } b \in B\} \times B$). Then $g * \psi = g$ since $\widehat{g * \psi}(\lambda, k) = \widetilde{g}(\lambda, k) \widehat{\psi}(\lambda) = \widetilde{g}(\lambda, k)$. Thus by the argument above, $\langle f, g \rangle = \langle f, g * \psi \rangle = 0$.

Thus it follows that \widetilde{f} is supported on a subset of $[\beta, \alpha] \times B$. We shall now show that it is not supported in a smaller annulus. We define

$$R_f^+ = \sup\{\lambda^2 + \rho^2 \mid (\lambda, b) \in \text{Suppt } \widetilde{f}\}, \quad R_f^- = \inf\{\lambda^2 + \rho^2 \mid (\lambda, b) \in \text{Suppt } \widetilde{f}\}.$$

Above we have proved that $c_1 \geq R_f^+$ and $1/c_2 \leq R_f^-$. Now we shall show that given any $\epsilon > 0$, $c_1 < R_f^+ + \epsilon$ and $1/c_2 > R_f^- - \epsilon$. For this we fix an $\epsilon > 0$. We take a $\psi \in C^2(G//K)$ such that $\widehat{\psi}$ is compactly supported, $\widehat{\psi} \equiv 1$ on the support of \widetilde{f} (hence $\text{Suppt } \widetilde{f} \subseteq \text{Suppt } \widehat{\psi}$) and $R_f^+ < R_\psi^+ < R_f^+ + \epsilon$, $R_f^- - \epsilon < R_\psi^- < R_f^-$. Then $\widehat{\psi} \widetilde{f} = \widetilde{f}$ and hence $f = f * \psi$. Thus by Proposition 3.2 (iv) and isomorphism of $C^2(G//K)$ and $C^2(\widehat{G//K})$, there exist seminorms ν of $C^2(X)$ and μ of $C^2(\widehat{X})$ such that

$$\|\Delta^n f\|_{2,\infty} = \|\Delta^n f * \psi\|_{2,\infty} = \|f * \Delta^n \psi\|_{2,\infty} \leq \|f\|_{2,\infty} \nu(\Delta^n \psi) \leq \|f\|_{2,\infty} \mu(\widehat{\Delta^n \psi}).$$

Thus,

$$\|\Delta^n f\|_{2,\infty} \leq \|f\|_{2,\infty} \mu((\lambda^2 + \rho^2)^n \widehat{\psi}) \leq \|f\|_{2,\infty} (R_\psi^+)^n n! C_{\psi,\mu}$$

for some finite constant $C_{\psi,\mu}$ which depends on ψ and μ . This implies

$$c_1 = \lim_{n \rightarrow \infty} \|\Delta^n f\|^{1/n} \leq R_\psi^+ < R_f^+ + \epsilon.$$

Replacing $\Delta^n f$ by $\Delta^{-n} f$ in the argument above, we get similarly,

$$\|\Delta^{-n} f\|_{2,\infty} \leq \|f\|_{2,\infty} (R_\psi^-)^{-n} n! C_{\psi',\mu}$$

which implies $c_2 = \lim_{n \rightarrow \infty} \|\Delta^{-n} f\|^{1/n} \leq (R_\psi^-)^{-1}$, hence $1/c_2 \geq R_\psi^- \geq R_f^- - \epsilon$. This completes the proof of part (b)

(c) If $c_1 c_2 = 1$ then $\alpha = \beta$, hence \tilde{f} is supported on the sphere $\{\alpha\} \times B$ of radius α . Therefore (c) follows from Lemma 4.10.

(a) We have used the two conditions of the hypothesis independently to prove that \tilde{f} is supported in a subset of $[0, \alpha]$ and also in a subset of $[\beta, \infty)$ for $\alpha, \beta \in \mathbb{R}^+$. If $\alpha < \beta$ then the support of \tilde{f} is empty and hence $f = 0$, contradicting the hypothesis. Therefore $\alpha \geq \beta$, equivalently $c_1 c_2 \geq 1$.

(d) When $\beta = 0$ equivalently $c_2 = 1/\rho^2$ then the annulus \mathbb{A}_β^α obviously reduces to a ball around origin of radius α . If $c_1 = \rho^2$, then $\alpha = 0$ implying that $\alpha = \beta = 0$ by (a) and the interval $[\beta, \alpha]$ degenerates to a singleton set $\{0\}$. Hence by (c) f is an eigenfunction with eigenvalue $-\rho^2$. Then by Proposition 3.2 (v), $f \notin L^{2,\infty}(X)$ contradicting the hypothesis. Therefore $c_1 > \rho^2$, equivalently $\alpha > 0$, i.e. the \mathbb{A}_β^α does not collapse to origin. \square

We shall now prove the lemmas to complete the proof of (c). We shall write ∂_λ , ∂_λ^n respectively for $\frac{d}{d\lambda}$ and $\frac{d^n}{d\lambda^n}$.

Lemma 4.5. *For any nonconstant polynomial P and $\lambda_0 > 0$, $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \notin L^{2,\infty}(X)$.*

Proof. Let P be a polynomial of degree n given by $P(y) = a_0 y^n + a_1 y^{n-1} + \dots + a_n$, $a_0 \neq 0$. We shall show that if $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$, then $\partial_\lambda \phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. Use of Lemma 4.6 then completes the proof.

So, we assume that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. If $n = 1$, then $P(\partial_\lambda)\phi_\lambda = a_0 \partial_\lambda \phi_\lambda + a_1 \phi_\lambda$. Since $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$ and $a_1 \phi_{\lambda_0} \in L^{2,\infty}(X)$ (see Proposition 3.2 (vii)) we have $a_0 \partial_\lambda \phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. If $n \geq 2$, we take a function $\psi \in C^2(G//K)$ such that $\hat{\psi}$ and its derivatives of orders up to $(n-2)$ are zero at λ_0 . Then $\partial_\lambda^{n-r}(\hat{\psi}(\lambda)\phi_\lambda)|_{\lambda=\lambda_0} = 0$ for all $2 \leq r \leq n$.

We note that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi = P(\partial_\lambda)(\phi_\lambda * \psi)|_{\lambda=\lambda_0}$ where the convolution can be justified from the estimate of $P(\partial_\lambda)\phi_\lambda$ (see (2.4), (2.5)). Hence,

$$P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi = \{a_0 \partial_\lambda^n(\hat{\psi}(\lambda)\phi_\lambda) + a_1 \partial_\lambda^{n-1}(\hat{\psi}(\lambda)\phi_\lambda)\}|_{\lambda=\lambda_0}.$$

Expanding the derivatives in the right hand side by Leibnitz rule and using that $\hat{\psi}$ and its derivatives of order $1, 2, \dots, n-2$ vanish at λ_0 we get,

$$P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi = [a_0 \{\phi_\lambda \partial_\lambda^n(\hat{\psi}(\lambda)) + n(\partial_\lambda \phi_\lambda) \partial_\lambda^{n-1}(\hat{\psi}(\lambda))\} + a_1 \phi_\lambda \partial_\lambda^{n-1}(\hat{\psi}(\lambda))]|_{\lambda=\lambda_0}.$$

The assumption $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$ implies that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi \in L^{2,\infty}(X)$ (Proposition 3.2 (iv)). Since $\phi_{\lambda_0} \in L^{2,\infty}(X)$ (Proposition 3.2 (vii)), we get from above that $\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. \square

Lemma 4.6. *For any $\lambda_0 \in \mathbb{R}^+$, $\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0} \notin L^{2,\infty}(X)$.*

Proof. In view of the polar decomposition and the corresponding integral formula (2.3) and the identification of A with \mathbb{R} , it suffices to show that $\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0}$ restricted to $[1, \infty)$ does not belong to $L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. We shall use the facts that $e^{-\rho t} \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$ and $te^{-\rho t} \notin L^{2,\infty}([1, \infty), e^{2\rho t} dt)$, which are easily verifiable through straightforward computation. We recall that ϕ_λ for any $\lambda \in \mathbb{R}^\times$, has the following expansion (see [37, 23])

$$\phi_\lambda(t) = e^{-\rho t} [c(\lambda)e^{i\lambda t} + c(-\lambda)e^{-i\lambda t} + E(\lambda, t)],$$

where

$$E(\lambda, t) = c(\lambda)e^{i\lambda t} \sum_{k=1}^{\infty} \Gamma_k(\lambda)e^{-2kt} + c(-\lambda)e^{-i\lambda t} \sum_{k=1}^{\infty} \Gamma_k(-\lambda)e^{-2kt}$$

and Γ_k are recursively defined by $\Gamma_0(\lambda) = 1$ and

$$(k+1)(k+1-i\lambda)\Gamma_{k+1} = (\rho+k)(\rho+k-i\lambda)\Gamma_k + m_{2\gamma} \sum_{j=0}^k (-1)^{k+j+1} (\rho+2j-i\lambda)\Gamma_j.$$

For $t \geq 1$ the series defining $E(\lambda, t)$ and its λ -derivative at $\lambda = \lambda_0$ are uniformly convergent. Term by term differentiation shows that $|E(\lambda, t)| \leq C_\lambda$ for some constant C_λ for $t \geq 1$. Thus $e^{-\rho t} E(\lambda, t) \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. Therefore we need to show that

$$e^{-\rho t} \partial_\lambda [c(\lambda)e^{i\lambda t} + c(-\lambda)e^{-i\lambda t}]|_{\lambda=\lambda_0} \notin L^{2,\infty}([1, \infty), e^{2\rho t} dt).$$

Noting that $\overline{c(\lambda)} = c(-\lambda)$ and writing $c(\lambda) = a(\lambda) + i b(\lambda)$ where $a(\lambda), b(\lambda)$ are real functions, we have

$$\begin{aligned} e^{-\rho t} \partial_\lambda [c(\lambda)e^{i\lambda t} + c(-\lambda)e^{-i\lambda t}] &= 2e^{-\rho t} \partial_\lambda (\Re(c(\lambda)e^{i\lambda t})) \\ &= 2e^{-\rho t} \partial_\lambda (a(\lambda) \cos \lambda t - b(\lambda) \sin \lambda t) \\ &= -2te^{-\rho t} (a(\lambda) \sin \lambda t + b(\lambda) \cos \lambda t) \\ &\quad + 2e^{-\rho t} (\partial_\lambda(a(\lambda)) \cos \lambda t - \partial_\lambda(b(\lambda)) \sin \lambda t). \end{aligned}$$

Since at $\lambda = \lambda_0$ the last term in the equality above is in $L^{2,\infty}([1, \infty), e^{2\rho t} dt)$, we need only to show that $g(t) = te^{-\rho t} (a(\lambda_0) \sin \lambda_0 t + b(\lambda_0) \cos \lambda_0 t) \notin L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. For the sake of meeting a contradiction, we assume that $g \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. Then its translation by $\pi/2\lambda_0$ is $g(t + \pi/2\lambda_0) = C(t + \pi/2\lambda_0)e^{-\rho t} (a(\lambda_0) \cos \lambda_0 t - b(\lambda_0) \sin \lambda_0 t)$, which is also in $L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. This follows from interpolation of the facts that for $1 < p < 2 < q$, translation by a fixed element in \mathbb{R} is a bounded operator from L^p to L^p and from L^q to L^q in the measure space $([1, \infty), e^{2\rho t} dt)$.

We note that the part $C(\pi/2\lambda_0)e^{-\rho t}(a(\lambda_0)\cos\lambda_0 t - b(\lambda_0)\sin\lambda_0 t)$ of $g(t + \pi/2\lambda_0)$ is in $L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. Therefore the other part of $g(t + \pi/2\lambda_0)$, given by $h(t) = te^{-\rho t}(-b(\lambda_0)\sin\lambda_0 t + a(\lambda_0)\cos\lambda_0 t) \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$. Since $g(t)$ and $h(t)$ are in $L^{2,\infty}([1, \infty), e^{2\rho t} dt)$ we have,

$$\begin{aligned} b(\lambda_0)g(t) + a(\lambda_0)h(t) &= te^{-\rho t}(a(\lambda_0)^2 + b(\lambda_0)^2)\cos\lambda_0 t \in L^{2,\infty}([1, \infty), e^{2\rho t} dt), \\ a(\lambda_0)g(t) - b(\lambda_0)h(t) &= te^{-\rho t}(a(\lambda_0)^2 + b(\lambda_0)^2)\sin\lambda_0 t \in L^{2,\infty}([1, \infty), e^{2\rho t} dt). \end{aligned}$$

Hence $(a(\lambda_0)^2 + b(\lambda_0)^2)e^{i\lambda_0 t}te^{-\rho t} \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$ which amounts to say that $te^{-\rho t} \in L^{2,\infty}([1, \infty), e^{2\rho t} dt)$, a contradiction. \square

Lemma 4.7. *For any polynomial P in one variable and $\xi \in \mathbb{R}$, $\mathcal{A}^*(P(\partial_\xi)e^{-i\xi t}) = P(\partial_\xi)\phi_\xi$ as L^2 -tempered distribution on X , equivalently $(\mathcal{A}^*)^{-1}(P(\partial_\xi)\phi_\xi) = P(\partial_\xi)e^{-i\xi t}$ as tempered distribution on \mathbb{R} .*

Proof. Enough to show this for $P(\partial_\xi) = \partial_\xi$. Let $\psi \in C^2(G//K)$. Then $\mathcal{A}\psi \in S(\mathbb{R})_{\text{even}}$. We have

$$\langle \mathcal{A}\psi, \partial_\xi e^{-i\xi t} \rangle = \langle \psi, \mathcal{A}^*(\partial_\xi e^{-i\xi t}) \rangle.$$

On the other hand using slice-projection theorem (see subsection 2.3.5) we have,

$$\langle \mathcal{A}\psi, \partial_\xi e^{-i\xi t} \rangle = \partial_\xi \mathcal{F}(\mathcal{A}\psi)(\xi) = \partial_\xi \widehat{\psi}(\xi) = \partial_\xi \langle \psi, \phi_\xi \rangle = \langle \psi, \partial_\xi \phi_\xi \rangle.$$

Thus $\langle \psi, \mathcal{A}^*(\partial_\xi e^{-i\xi t}) \rangle = \langle \psi, \partial_\xi \phi_\xi \rangle$, for all $\psi \in C^2(G//K)$, implying $\mathcal{A}^*(\partial_\xi e^{-i\xi t}) = \partial_\xi \phi_\xi$ as L^2 -tempered distributions. As \mathcal{A}^* is an isomorphism from $S(\mathbb{R})_{\text{even}}$ to $C^2(G//K)$, the equivalent statement follows. \square

Lemma 4.8. *Let f_1, f_2 be two nonzero functions in $L^{2,\infty}(X)$. Then the following statements are true.*

- (a) *There exists $x \in G$ such that $R(\ell_x f_1) \neq 0$.*
- (b) *If for some $x \in G$, $R(\ell_x f_1) \neq 0$, then $R(\Delta^n \ell_x f_1) \neq 0$ for all $n \in \mathbb{Z}$.*
- (c) *If $R(\ell_x f_1) = R(\ell_x f_2)$ for all $x \in G$, then $f_1 = f_2$.*

Proof. If $R(\ell_x f_1) = 0$ for all $x \in G$, then for any $h \in C^2(G//K)$, $\langle \ell_x f_1, h \rangle = 0$. Let $h_t, t > 0$ be the heat kernel which is an element in $C^2(G//K)$ defined through its spherical Fourier transform $\widehat{h_t}(\lambda) = e^{-t(\lambda^2 + \rho^2)}$. Taking $h = h_t$ we thus get $\langle \ell_x f_1, h_t \rangle = 0$, i.e. $f_1 * h_t \equiv 0$ for all $t > 0$. But $f_1 * h_t \rightarrow f_1$ as $t \rightarrow 0$ in the sense of distribution. Therefore $f_1 = 0$ which contradicts that f_1 is nonzero. This proves (a). Applying this on $f_1 - f_2$ we get (c).

For (b) it is enough to show that $R(\ell_x f_1) \neq 0$ implies that $\Delta^{-1}R(\ell_x f_1) \neq 0$ and $\Delta R(\ell_x f_1) \neq 0$. Indeed $\Delta^{-1}R(\ell_x f_1) = 0$ implies $R(\ell_x f_1) = \Delta \Delta^{-1}R(\ell_x f_1) = 0$. On the other hand if $\Delta R(\ell_x f_1) = 0$, then $\langle \Delta R(\ell_x f_1), \psi \rangle = 0$ and hence

$\langle R(\ell_x f_1), \Delta \psi \rangle = 0$ for all $\psi \in C^2(G//K)$. Since for any $\phi \in C^2(G//K)$, $\widehat{\phi}(\lambda)(\lambda^2 + \rho^2)^{-1} \in C^2(\widehat{G//K})$ (see subsection 2.3.4), ϕ can be written as $\phi = \Delta \psi$ for some $\psi \in C^2(G//K)$. Thus $\langle R(\ell_x f_1), \phi \rangle = 0$ for any $\phi \in C^2(G//K)$, i.e. $R(\ell_x f_1) = 0$. \square

Lemma 4.9. *Suppose that the support of the (distributional) spherical Fourier transform \widehat{f} of $f \in L^{2,\infty}(G//K)$ is $\{\alpha\}$ for some $\alpha > 0$. Then $f = c\phi_\alpha$ for some constant c .*

Proof. Since f is an L^2 -tempered distribution (Proposition 3.2 (ii)), $(\mathcal{A}^*)^{-1}f$ is an even tempered distribution on \mathbb{R} (see subsection 2.3.5). We recall that $C^2(\widehat{G//K}) = S(\mathbb{R})_{\text{even}}$. The Euclidean Fourier transform of $(\mathcal{A}^*)^{-1}f$ in the sense of distribution denoted by $\mathcal{F}((\mathcal{A}^*)^{-1}f)$ is same as the spherical Fourier transform of f in the sense of L^2 -tempered distribution denoted by \widehat{f} . Indeed, we take $\phi, \psi \in S(\mathbb{R})_{\text{even}}$ such that $\mathcal{F}(\psi) = \phi$. As Abel transform is an isomorphism between $C^2(G//K)$ and $S(\mathbb{R})_{\text{even}}$, there is $g \in C^2(G//K)$ such that $\mathcal{A}g = \psi$, hence by slice-projection theorem $\widehat{g} = \mathcal{F}(\psi)$. Then we have

$$\begin{aligned} \langle \mathcal{F}((\mathcal{A}^*)^{-1}f), \phi \rangle &= \langle (\mathcal{A}^*)^{-1}f, \psi \rangle = \langle (\mathcal{A}^*)^{-1}f, \mathcal{A}g \rangle \\ &= \langle \mathcal{A}^*(\mathcal{A}^*)^{-1}f, g \rangle = \langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \\ &= \langle \widehat{f}, \mathcal{F}(\psi) \rangle = \langle \widehat{f}, \phi \rangle. \end{aligned}$$

Thus $\langle \mathcal{F}((\mathcal{A}^*)^{-1}f), \phi \rangle = \langle \widehat{f}, \phi \rangle$ where in the left hand side ϕ is interpreted as a function of $S(\mathbb{R})_{\text{even}}$ and on the right hand side ϕ is an element of $C^2(\widehat{G//K})$. Therefore $\mathcal{F}((\mathcal{A}^*)^{-1}f)$ is supported on $\{\alpha\}$.

Therefore by [35, Theorem 6.25],

$$(\mathcal{A}^*)^{-1}f(t) = [P_1(\partial_\lambda)e^{i\lambda t} + P_2(\partial_\lambda)e^{-i\lambda t}]|_{\lambda=\alpha}$$

for two polynomials P_1 and P_2 .

As $\phi_\lambda = \phi_{-\lambda}$ we have by Lemma 4.7, $f = P(\partial_\lambda)\phi_\lambda|_{\lambda=\alpha}$ for some polynomial P . Since $f \in L^{2,\infty}(X)$, by Lemma 4.5 the polynomial is constant. Hence $f = c\phi_\alpha$ for some constant c . \square

Lemma 4.10. *Suppose that for a function $f \in L^{2,\infty}(X)$, \widehat{f} is supported on the sphere of radius $\alpha > 0$ in $\mathbb{R}^+ \times B$. Then $f = \mathcal{P}_\alpha F$ for some $F \in L^2(B)$.*

Proof. By Observation 4.4, for any $x \in G$ either $R(\ell_x f)$ is zero or its spherical Fourier transform is supported on $\{\alpha\}$. We also note that since $f \in L^{2,\infty}(X)$, $R(\ell_x f) \in L^{2,\infty}(G//K)$. Therefore by Lemma 4.9, $\Delta R(\ell_x f) = -(\alpha^2 + \rho^2)R(\ell_x f)$ for all $x \in G$. That is $R(\ell_x \Delta f) = R(\ell_x [-(\alpha^2 + \rho^2)f])$ for all $x \in G$. Hence by

Lemma 4.8 (c) $\Delta f = -(\alpha^2 + \rho^2)f$. Since $f \in L^{2,\infty}(X)$, by Proposition 3.2 (viii), we have $f = \mathcal{P}_\alpha u$ for some $u \in L^2(B)$. \square

5. CONCLUDING REMARKS

(1) As mentioned earlier, Lemma 4.10 may be considered as an independent result. We have the following generalization. See [22, pp. 205], [40, Lemma 2.2] for Euclidean results of this genre.

Proposition 5.1. *Suppose that a locally integrable function f on X satisfies $f(x)(1+|x|)^{-M} \in L^{2,\infty}(X)$ for some fixed nonnegative integer M and \tilde{f} is supported on the sphere $\{\alpha\} \times B$ of radius $\alpha > 0$ in $\mathbb{R}^+ \times B$. Then $(\Delta + \alpha^2 + \rho^2)^{M+1}f = 0$, i.e. f is a generalized eigenfunction of Δ with eigenvalue $-(\alpha^2 + \rho^2)$. In particular if $M = 0$ then f is an eigenfunction.*

We include a sketch of the proof.

Proof. We have $\ell_x(f(y)/(1+|y|)^M) = \ell_x f(y)/(1+|x^{-1}y|) \in L^{2,\infty}(X)$. Since $(1+|xy|) < (1+|x|)(1+|y|)$ ([15, Prop. 4.6.11], $\ell_x f(y)/(1+|y|) \in L^{2,\infty}(X)$. Now as $R(\ell_x f)(y)/(1+|y|)^M = R(\ell_x f)(y)/(1+|y|)^M$, we have $R(\ell_x f)(y)/(1+|y|)^M \in L^{2,\infty}(G//K)$. Thus $R(\ell_x f)$ is a L^2 -tempered distribution. By Observation 4.4, if for some $x \in G$, $R(\ell_x f) \neq 0$ then $\widehat{R\ell_x f}$ is supported on $\{\alpha\}$. We fix $x \in G$, such that $R(\ell_x f) \neq 0$. Proceeding as the proof of Lemma 4.9 we conclude that $R(\ell_x f) = P_x(\partial_\lambda)\phi_\lambda|_{\lambda=\alpha}$ where the polynomial P_x depends on $x \in G$. Hence by Lemma 5.2 proved below, $(\Delta + \alpha^2 + \rho^2)^{\deg P_x + 1}R(\ell_x f) = 0$. However, the condition $R(\ell_x f)/(1+|\cdot|)^M \in L^{2,\infty}(G//K)$ puts an upper bound for the degree of polynomial, precisely $\deg P_x \leq M$ as can be proved going through the steps similar to Lemma 4.6. Thus for all $x \in G$, $(\Delta + \alpha^2 + \rho^2)^{M+1}R(\ell_x f) = 0$. That is $R(\ell_x(\Delta + \alpha^2 + \rho^2)^{M+1}f) = 0$ and hence by Lemma 4.8, $(\Delta + \alpha^2 + \rho^2)^{M+1}f = 0$. \square

Now we shall prove the lemma used in the proposition above.

Lemma 5.2. *If e_λ is an eigenfunction of Δ with eigenvalue $A(\lambda)$ then for any polynomial P in one variable of degree $m \in \mathbb{N}$, $(\Delta - A(\lambda))^{m+1}P(\partial_\lambda)e_\lambda = 0$ i.e. $P(\partial_\lambda)e_\lambda$ is a generalized eigenfunction of Δ with eigenvalue $A(\lambda)$.*

Proof. It suffices to show that $(\Delta - A(\lambda))^{m+1}\partial_\lambda^m e_\lambda = 0$, which can be verified by straightforward computation for $m = 1, 2$. Then we use induction. Suppose the result is true for $m = 1, 2, \dots, n-1$. Now,

$$(\Delta - A(\lambda))^{n+1}\partial_\lambda^n e_\lambda = (\Delta - A(\lambda))^n[\partial_\lambda^n(A(\lambda)e_\lambda) - A(\lambda)\partial_\lambda^n e_\lambda].$$

Expanding the part in square bracket [...] in the right hand side above by Leibnitz rule we see that each term in it is of the form $C\partial_\lambda^r A(\lambda)\partial_\lambda^{n-r} e_\lambda$ for $r = 1, 2, \dots, n$. From induction hypothesis it follows that $(\Delta - A(\lambda))^n \partial_\lambda^{n-r} e_\lambda = 0$. This completes the proof. \square

Proposition 5.1 vindicates a generalization of Theorem 1.1. For a fixed $M > 0$ we define a weighted norm $\|\cdot\|_M$ in the following way. For measurable function f on X , let $g(x) = f(x)(1 + |x|)^{-M}$. Then $\|f\|_M = \|g\|_{2,\infty}$.

Theorem 5.3. *Let f be a nonzero measurable function on X with $\|f\|_M < \infty$. Suppose for constants $c_1 \geq \rho^2, c_2 \leq 1/\rho^2$*

$$\lim_{n \rightarrow \infty} \|\Delta^n f\|_M^{1/n} = c_1, \quad \lim_{n \rightarrow \infty} \|\Delta^{-n} f\|_M^{1/n} = c_2.$$

Let $\beta = \sqrt{1/c_2 - \rho^2}$ and $\alpha = \sqrt{c_1 - \rho^2}$. Then we have conclusions (a) and (b) of Theorem 1.1, while (c) and (d) of that theorem are replaced by

- (c) *If $c_1 c_2 = 1$ then f is a generalized eigenfunction with eigenvalue $-c_1$,*
- (d) *The annulus \mathbb{A}_β^α may reduce to a ball around origin and may also collapse to the origin.*

Proof of Theorem 1.1 can be easily adapted to prove this, which we omit for brevity. We only note that under the norm-condition here which is more relaxed than that of Theorem 1.1, this theorem allows collapsing of the annulus to the origin (see (d) above). This corresponds to the case $c_1 = 1/c_2 = \rho^2$, hence $c_1 c_2 = 1$ and thus is a subcase of (c). Precisely in this case f is a generalized eigenfunction of Δ with eigenvalue $-\rho^2$, a particular case of which is ϕ_0 .

(2) Given the similarity of the setting, it is not surprising that our line of argument sometimes goes near the study of real inverse Paley-Wiener theorems and the characterization of eigenfunctions of Δ mentioned earlier. We pause briefly to point out the distinguishing features of our study. In [1] Andersen considered real inverse Paley-wiener theorem characterizing functions in $L^2(X)$ whose Fourier transform is supported in a ball around origin in $\mathbb{R}^+ \times B$. It is clear from the proof of Theorem 1.1 that only positive integral powers of Δ is required for this. As [1] is dealing with L^2 -functions, Plancherel theorem has a crucial presence in the proof. But as explained in Section 3, this precludes the possibility of the support to degenerate and allow the function to be an eigenfunction. On the other hand aim of [28, 33] is to obtain a characterization of the eigenfunction of Δ . However the hypothesis of those theorems are strong enough to determine the precise annulus around origin inside which \tilde{f} is supported.

(3) We recall that through the Iwasawa decomposition $G = NAK$, $X = G/K$ can be identified with the solvable Lie group $N \rtimes A$. Thus the rank one Riemannian symmetric spaces of noncompact type becomes a subclass of Damek-Ricci spaces (known also as NA groups). Indeed symmetric spaces are the most distinguished prototypes of NA groups, even though they account for a very thin subcollection (see [6]). In general a Damek-Ricci space is a Riemannian manifold and a solvable Lie group but not a symmetric space. To deal with a general Damek-Ricci space say S one faces many fresh difficulties. A major challenge is the absence of semisimple machinery which enters the picture through the G -action on $X = G/K$. A particular discomfort arises as we cannot decompose a function on S in K -types; a very useful tool while working on symmetric spaces. The sense of *radiality* in these spaces is not connected with group action. Keeping this in mind we have completely avoided such well-known techniques for symmetric spaces. The proof given here is thus readily extendable to harmonic NA groups. However for Damek-Ricci spaces we have to make a compromise, as the characterization of $L^{2,\infty}$ -eigenfunction as Poisson transform is still unavailable in the literature, albeit expected. Precisely (Theorem 1.1 (c)) ‘ f is a Poisson transform’ have to be substituted by a weaker statement ‘ f is an eigenfunction of Δ with eigenvalue $-c_1$ ’.

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